M.S. Advanced and Ph.D. Entrance Exams

Mathematics

Fall 2020

# Differential Equations

WVU Mathematics Department

# ODE ENTRANCE EXAM, FALL 2020

September 16 2020

*Solve all six problems. Show all work. Explain and justify your answers. All problems carry equal weight.*
*Notation:* $y' = \frac{dy}{dt}$

**Name** ______________________________ **Total Score** ________

**1.** Let $m, n \in \mathbb{N}$ and consider the differential system

$$x' = my^{2m-1},\ y' = -nx^{2n-1}. \tag{1}$$

($x, y$ are scalar functions of $t$). Show that $(0,0)$ is a stable equilibrium of (1).

**2.** Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$f(y) = \begin{bmatrix} \cos(y_1 y_2 y_3) \\ \sin(y_1 + y_2 + y_3) \\ \exp\left(\dfrac{2y_1 y_2}{1 + y_1^2 + y_2^2}\right) \end{bmatrix}$$

for all $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{R}^3$. Show that for the differential system

$$y' = f(y)$$

a) there are no critical points;
b) all solutions exist on $(-\infty, \infty)$.

**3**. Let $n > 1$ be a natural number.
a) Let $A$ be an $n \times n$ constant matrix. Prove that $Y(t) = \exp(tA)$ is a solution to the matrix differential equation $Y' = AY$.
b) Let $A(t)$ be an $n \times n$ time dependent matrix function of $t$ on $(-\infty, \infty)$. Must $Y(t) = \exp(tA(t))$ be a solution of $Y' = A(t)Y$? Explain.

**4**. Solve the differential equation

$$y' = t^2 y - 3t^2$$

**5**. Consider the ODE system

$$x' = y^3 - 4x, \; y' = y^3 - y - 3x.$$

a) Find all critical points and classify their stability.
b) Show that the line $x = y$ is invariant (any solution that starts on it stays on it).
c) If $(x(0), y(0)) = (-1, 1)$, find $x(t) - y(t)$.

**6.** Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(t) > f(t)$ for all $t \in \mathbb{R}$, and $f(0) = 0$. Show that $f(t) > 0$ for any $t > 0$.